Schubert Calculus and Threshold Polynomials of Affine Fusion

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Abstract

We show how the threshold level of affine fusion, the fusion of Wess-Zumino-Witten (WZW) conformal field theories, fits into the Schubert calculus introduced by Gepner. The Pieri rule can be modified in a simple way to include the threshold level, so that calculations may be done for all (non-negative integer) levels at once. With the usual Giambelli formula, the modified Pieri formula deforms the tensor product coefficients (and the fusion coefficients) into what we call threshold polynomials. We compare them with the q-deformed tensor product coefficients and fusion coefficients that are related to q-deformed weight multiplicities. We also discuss the meaning of the threshold level in the context of paths on graphs.

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1. Introduction

Gepner found geometrical and topological interpretations of the fusion rings of Wess-Zumino-Witten (WZW) conformal field theories [1]. He described them using a Schubert calculus, a "quantum version" of the classical Schubert calculus that is fundamental in the geometry and topology of complex manifolds (see [2], e.g.).

Gepner also pointed out a correspondence between the WZW fusion rings and the chiral rings of $N = 2$ superconformal theories. These two observations have been seminal. For example, their relation was clarified in [3], where the new Schubert calculus was shown to describe the quantum cohomology of Grassmannians. Also, the $N = 2$ interpretation led to new realisations of WZW fusion rings in topological theories [4][5][6].

We study the Schubert calculus of WZW fusion rings. Our initial motivation was computational. In Gepner's approach, a fusion potential is introduced whose derivatives give the fusion constraints to be implemented. The fusion potential, and so the fusion constraints, are level dependent. Therefore a significant part of any computation must be re-done whenever the level is changed. By the depth rule [7], however, the results are simpler than this procedure indicates. A *threshold level* exists for each coupling [8][9]; in any fusion product of two fixed fields, a third appears in the decomposition for all integer levels greater than or equal to a characteristic one†. Therefore, finding the threshold levels for a fixed product amounts to finding the fusion rules for all levels at once.

We show how to incorporate the notion of threshold level into Gepner's Schubert calculus for WZW fusion. This is done in section 3, after a review is given in section 2, where the notation is also established.

Another motivation for this work emerges in section 2: it is convenient to encode the threshold levels in generating polynomials, dubbed *threshold polynomials*. These are then polynomial deformations of tensor product coefficients and fusion coefficients. Similar objects, the quantum group (q-)deformations of tensor product coefficients [10][11][12][13] and fusion coefficients [14][15] have been studied previously. Most importantly to us, the q-deformed coefficients are related to the q-deformed weight multiplicities defined by Lusztig [16]. In section 4 we compare our deformations with the q-deformations. We show that the new deformations are related in a similar way to deformations of the weight multiplicities, that are natural from the point of view of a conjectured refinement [9][17][18] of the Gepner-Witten depth rule [7].

† To the best of our knowledge, however, a completely rigorous demonstration of the existence of threshold levels is still lacking.

As we have argued, the threshold level has computational advantages over the use of fusion potentials, and the relations derived from them. But again, the connection with geometry, topology and $N = 2$ superconformal theories was the point of [1], not computation. There the fusion potential played a central role. But one can't have it both ways: we indicate at the end of section 3 that a deformed fusion potential that incorporates the threshold levels cannot be written. Nevertheless, one might hope to give the threshold level a somewhat deeper motivation, perhaps through its meaning in the many different realisations of WZW fusion rings. In section 5 we make a very small start in this direction; we discuss the meaning of the threshold level in the context of paths on graphs (see [19][20], and references therein).

Section 6 is a short conclusion.

2. WZW fusion, threshold level, and threshold polynomials

Let us first establish notation. For the most part, we restrict attention to the simple Lie algebras A_r and the affine algebras $A_r^{(1)}$ that are the untwisted central extensions of their loop algebras. When the level k is fixed, we denote the affine algebra by $A_{r,k}$. However, we use a notation that is easily adapted to any untwisted affine Kac-Moody algebra $X_r^{(1)}$ (or $X_{r,k}$) based on a simple Lie algebra X_r, and expect that such generalisation is straightforward.

The set of roots of X_r will be written as R, and the set of positive (negative) roots as $R_>$ ($R_<$). If $\alpha \in R$ is a root, then the corresponding co-root is defined as $\alpha^\vee := 2\alpha/(\alpha,\alpha)$.

Let $F = \{\Lambda_1, \ldots, \Lambda_r\}$ denote the set of fundamental weights of X_r, and

$$P \ := \ \{ \lambda = \sum_{i=1}^{r} \lambda_i \Lambda_i \mid \lambda_i \in \mathbb{Z} \} \tag{2.1}$$

the set of integral weights. The set of dominant integral weights,

$$P_\geq \ := \ \{ \lambda = \sum_{i=1}^{r} \lambda_i \Lambda_i \mid \lambda_i \in \mathbb{Z}_\geq \} \ , \tag{2.2}$$

is the set of highest weights for irreducible integrable modules of X_r.

Let $M(\lambda)$ denote an irreducible module of X_r, of highest weight $\lambda \in P_\geq$. The set of weights of $M(\lambda)$ will be denoted $P(\lambda)$.

The irreducible integrable modules of $X_{r,k}$ have highest weights that project to the following set of dominant weights of X_r:

$$P_{\geq}^{k} \ := \ \{ \lambda = \sum_{i=1}^{r} \lambda_i \Lambda_i \mid \lambda_i \in \mathbb{Z}_{\geq}, \ \sum_{i=1}^{r} \lambda_i a_i^{\vee} \leq k \} \ . \tag{2.3}$$

The $a_i^{\vee}$ are the co-marks, defined by $a_0^{\vee} = 1$, and

$$\theta^{\vee} \ = \ \theta \ = \ \sum_{i=1}^{r} a_i^{\vee} \alpha_i^{\vee} \ , \tag{2.4}$$

where θ ($\theta^{\vee}$) denotes the highest (co-)root of X_r. We normalise $(\theta, \theta) = 2$.

The Weyl group of X_r will be denoted by W, and the shifted action of $w \in W$ on a weight λ by $w.\lambda = w(\lambda + \rho) - \rho$, where $\rho = \sum_{i=1}^{r} \Lambda_i = \sum_{\alpha \in R_>} \alpha/2$ is the Weyl vector. W^k will indicate the projection of the affine Weyl group, the Weyl group of $X_{r,k}$, onto the horizontal weight space, the weight space of X_r. W is generated by the primitive reflections r_i, $i = 1, \ldots, r$, with action

$$r_i \lambda \ = \ \lambda \ - \ (\lambda, \alpha_i^{\vee}) \alpha_i \tag{2.5}$$

on any weight λ. In order to enlarge W to W^k, we adjoin r_0 to the generating set. Its shifted action is

$$r_0.\lambda \ = \ r_{\theta}.\lambda + (k + x)\theta \ , \tag{2.6}$$

where x is the dual Coxeter number of X_r. Notice the k-dependence of the action of W^k on P, coming from that of r_0.

We write the decomposition of the tensor product of two irreducible integrable X_r-modules as

$$M(\lambda) \otimes M(\mu) \ = \ \bigoplus_{\nu \in P_{\geq}} T_{\lambda,\mu}^{\nu} \, M(\nu) \quad . \tag{2.7}$$

We will call the $T_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq}$ tensor product coefficients. We indicate the affine fusion of two modules of $X_{r,k}$ by writing the truncated tensor product of the corresponding modules $M(\lambda)$ and $M(\mu)$ of X_r:

$$M(\lambda) \otimes_k M(\mu) \ = \ \bigoplus_{\nu \in P_{\geq}^{k}} {}^{(k)}T_{\lambda,\mu}^{\nu} \, M(\nu) \quad . \tag{2.8}$$

The fusion coefficients obey

$${}^{(k)}T_{\lambda,\mu}^{\nu} \ \leq \ {}^{(k+1)}T_{\lambda,\mu}^{\nu} \ , \tag{2.9}$$

and furthermore

$$\lim_{k\to\infty} {}^{(k)}T_{\lambda,\mu}^{\nu} \;=\; T_{\lambda,\mu}^{\nu} \ . \tag{2.10}$$

We can encode the fusion products for all levels by including the threshold levels t as subscripts in the tensor product decomposition [8][9]. If we denote by $\mathcal{S}_t$ the operator that includes these subscripts, we can write

$$\mathcal{S}_t\,[M(\lambda)\otimes M(\mu)] \;=\; \bigoplus_{\nu\in P_{\geq}}\bigoplus_{t\in\mathbb{Z}_{\geq}} T_{\lambda,\mu}^{\nu(t)}\,M(\nu)_t \ . \tag{2.11}$$

Then

$${}^{(k)}T_{\lambda,\mu}^{\nu} \;=\; \sum_{t=0}^{k} T_{\lambda,\mu}^{\nu(t)} \ . \tag{2.12}$$

We call the fixed-threshold-level coefficients $T_{\lambda,\mu}^{\nu(t)}$, the *threshold coefficients.* For example, we can modify the A_2 tensor product to

$$\mathcal{S}_t\left[M(1,1)^{\otimes 2}\right] \;=\; M(2,2)_4\oplus M(3,0)_3\oplus M(0,3)_3\oplus 2M(1,1)_{2,3}\oplus M(0,0)_2\ , \tag{2.13}$$

encoding the corresponding fusions at all levels. Here $M(a,b) := M(\lambda)$, with $\lambda = a\Lambda_1+b\Lambda_2$, and $pM(a,b)_{t_1,\dots,t_p} := M(a,b)_{t_1}\oplus\cdots\oplus M(a,b)_{t_p}$.

From the notational point of view, the threshold levels are unnecessarily large numbers, since

$$T_{\lambda,\mu}^{\nu(t)} \;\neq\; 0 \quad\Rightarrow\quad t\;\geq\;(\nu,\theta)\ . \tag{2.14}$$

Consequently, one could also define the *threshold delay* d by

$$d \;:=\; t-(\nu,\theta)\ . \tag{2.15}$$

Writing the delays as superscripts, the right-hand side of (2.13) is replaced by

$$M(2,2)^2\oplus M(3,0)^0\oplus M(0,3)^0\oplus 2M(1,1)^{0,1}\oplus M(0,0)^2\ . \tag{2.16}$$

This is a minor point, so we'll stick to using the threshold levels. In section 4, however, (2.15) will reappear.

As (2.13) makes clear, we need to consider N-tuples of threshold levels. It is convenient to encode them in *threshold polynomials*, defined by

$$T_{\lambda,\mu}^{\nu}[\ell] \;:=\; \sum_{t\in\mathbb{Z}_{\geq}} \ell^{\,t}\,T_{\lambda,\mu}^{\nu(t)} \ . \tag{2.17}$$

Then

$$\mathcal{L}^t\left[M(\lambda)\otimes M(\mu)\right] \;=\; \bigoplus_{\nu\in P_\geq} T^\nu_{\lambda,\mu}[\ell]\, M(\nu) \tag{2.18}$$

is equivalent to (2.11). For example, the A_2 tensor product (2.13) is rewritten as

$$\mathcal{L}^t\left[M(1,1)^{\otimes 2}\right] \;=\; \ell^4\, M(2,2)\oplus \ell^3\, M(3,0)\oplus \ell^3\, M(0,3)\oplus (\ell^2+\ell^3)\, M(1,1)\oplus \ell^2\, M(0,0)\ . \tag{2.19}$$

The threshold polynomials can be regarded as deformations of the tensor product coefficients, since

$$T^\nu_{\lambda,\mu}[1] \;=\; T^\nu_{\lambda,\mu}\ , \tag{2.20}$$

so that the tensor product (2.7) is recovered when $\ell = 1$. Furthermore, we define the deformation of the fusion coefficient as

$${}^{(k)}T^\nu_{\lambda,\mu}[\ell] \;:=\; \sum_{t=0}^{k} \ell^t T^{\nu(t)}_{\lambda,\mu}\ . \tag{2.21}$$

So ${}^{(k)}T^\nu_{\lambda,\mu}[\ell]$ is the degree$\leq k$ part of the polynomial $T^\nu_{\lambda,\mu}[\ell]$, and

$${}^{(k)}T^\nu_{\lambda,\mu}[1] \;=\; {}^{(k)}T^\nu_{\lambda,\mu}\ . \tag{2.22}$$

(2.10) is deformed to

$$\lim_{k\to\infty}\ {}^{(k)}T^\nu_{\lambda,\mu}[\ell] \;=\; T^\nu_{\lambda,\mu}[\ell]\ , \tag{2.23}$$

by construction.

From (2.17), we see that the threshold polynomials are the generating functions for the threshold coefficients. Consequently, they are related to the generating functions for fusion rules studied in [8], where the threshold level was first introduced (but named later in [9]). For completeness, we indicate the relation here.

The generating function for fusion rules is defined as

$$G(L,M,N;d) \;:=\; \sum_{\lambda,\mu,\nu\in P_\geq}\ \sum_{k=0}^{\infty} {}^{(k)}T_{\lambda,\mu,\nu}\, d^k\, L^\lambda M^\mu N^\nu\quad , \tag{2.24}$$

where the dummy variables L, M, N satisfy $L^\lambda L^{\lambda'} = L^{\lambda+\lambda'}$, etc. Here ${}^{(k)}T_{\lambda,\mu,\nu} := {}^{(k)}T^{C\nu}_{\lambda,\mu}$, where $C\nu$ is the highest weight of the module conjugate to $M(\nu)$. Using (2.12) and switching the order of summations, we arrive at

$$G(L,M,N;d) \;=\; (1-d)^{-1} \sum_{\lambda,\mu,\nu\in P_\geq} L^\lambda M^\mu N^\nu\, T_{\lambda,\mu,\nu}[d]\ . \tag{2.25}$$

Here $T_{\lambda,\mu,\nu}[d] = \sum_{t=0}^{\infty} d^t\, T^{C\nu}_{\lambda,\mu}[d]$; see (2.21) and (2.23). Hence the only difference between the generating functions for deformed tensor product coefficients and fusion rules is $(1-d)^{-1}$, a factor characteristic of the existence of a threshold level [8].

For successive fusions, we need a memory of the threshold levels. For example, suppose we need to calculate $\mathcal{S}_t\left[M(\phi)\otimes(M(1,1)^{\otimes 2})\right]$. Then using (2.13), we would encounter products like $\mathcal{S}_t\left[M(\phi)\otimes M(1,1)_3\right]$. So (2.11) is only a special case of what we need. Abusing notation slightly, we attach threshold levels to the factor modules in the tensor products, and write

$$M(\lambda)_r \otimes M(\mu)_s \;=\; \bigoplus_{\nu\in P_\geq}\bigoplus_{t\in\mathbb{Z}_\geq} T^{\nu(t)}_{\lambda(r),\mu(s)}\, M(\nu)_t \quad . \tag{2.26}$$

For example, we have

$$M(1,1)_2 \otimes M(1,1)_3 \;=\; M(2,2)_4 \oplus M(3,0)_3 \oplus M(0,3)_3 \oplus 2M(1,1)_{3,3} \oplus M(0,0)_3 \;. \tag{2.27}$$

(2.11) is recovered from (2.26) by setting $r=(\lambda,\theta)$ and $s=(\mu,\theta)$.

Again using polynomials to carry the N-tuples of threshold levels, we write

$$\mathcal{L}^t\left[M(\lambda)_r \,\otimes\, M(\mu)_s\right] \;=\; \bigotimes_{\nu\in P_\geq} T^{\nu}_{\lambda(r),\mu(s)}[\ell]\, M(\nu) \;. \tag{2.28}$$

Comparing (2.27) with (2.13), for example, shows that there is a simple relation between the coefficients $T^{\nu(t)}_{\lambda(r),\mu(s)}$ and $T^{\nu(t)}_{\lambda,\mu}$. To write it in polynomial form, we define

$$\ell^a \circ \ell^b \;:=\; \ell^{\,\max\{a,b\}} \;. \tag{2.29}$$

and extend bilinearly (so that two polynomials can be multiplied). Then we have

$$T^{\nu}_{\lambda(r),\mu(s)}[\ell] \;=\; \ell^r \circ \ell^s \circ T^{\nu}_{\lambda,\mu}[\ell] \;. \tag{2.30}$$

We see that the polynomials $T^{\nu}_{\lambda,\mu}[\ell]$ are fundamental, and so we will concentrate on them henceforth.

The definition (2.29), however, is natural from the point of view of threshold polynomials. With it, we can generalise the crossing symmetry of the tensor product coefficients,

$$\sum_{\zeta\in P_\geq} T^{\zeta}_{\lambda,\mu}\, T^{\nu}_{\zeta,\varphi} = \sum_{\zeta\in P_\geq} T^{\nu}_{\lambda,\zeta}\, T^{\zeta}_{\mu,\varphi} \;, \tag{2.31}$$

to

$$\sum_{\zeta\in P_\geq} T^{\zeta}_{\lambda,\mu}[\ell] \circ T^{\nu}_{\zeta,\varphi}[\ell] = \sum_{\zeta\in P_\geq} T^{\nu}_{\lambda,\zeta}[\ell] \circ T^{\zeta}_{\mu,\varphi}[\ell] \ . \tag{2.32}$$

Furthermore, the crossing symmetry for the fusion coefficients

$$\sum_{\zeta\in P^k_\geq} {}^{(k)}T^{\zeta}_{\lambda,\mu}\ {}^{(k)}T^{\nu}_{\zeta,\varphi} = \sum_{\zeta\in P^k_\geq} {}^{(k)}T^{\nu}_{\lambda,\zeta}\ {}^{(k)}T^{\zeta}_{\mu,\varphi} \tag{2.33}$$

deforms to

$$\sum_{\zeta\in P^k_\geq} {}^{(k)}T^{\zeta}_{\lambda,\mu}[\ell] \circ {}^{(k)}T^{\nu}_{\zeta,\varphi}[\ell] = \sum_{\zeta\in P^k_\geq} {}^{(k)}T^{\nu}_{\lambda,\zeta}[\ell] \circ {}^{(k)}T^{\zeta}_{\mu,\varphi}[\ell] \ . \tag{2.34}$$

3. Schubert calculus, threshold level, and threshold polynomials

The Schubert calculus is based on the Pieri and Giambelli formulas. For discussions of them emphasising the geometric context see [2][1]. More relevant to us is their use in representation theory; consult [21], e.g.

The Pieri formula is

$$T^{\nu}_{\lambda,\Lambda} \ = \begin{cases} 1 \ , & \text{if } \nu-\lambda\in P(\Lambda) \ ; \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{3.1}$$

where Λ is a fundamental weight, i.e. $\Lambda\in F$. Here we are specialising to the algebras A_r, although the formulas for other algebras are only slightly more complicated.

Adapted to include threshold polynomials, the Pieri formula simply becomes

$$T^{\nu}_{\lambda,\Lambda}[\ell] \ = \begin{cases} \ell^{\,(\lambda,\theta)} \circ \ell^{\,(\nu,\theta)} \ , & \text{if } \nu-\lambda\in P(\Lambda) \ ; \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{3.2}$$

Fundamental monomials

$$M(\Lambda^\mu) \ := \ M(\Lambda_1)^{\otimes\mu_1}\otimes M(\Lambda_2)^{\otimes\mu_2}\otimes\cdots\otimes M(\Lambda_r)^{\otimes\mu_r} \ = \ \bigotimes_{i=1}^{r} M(\Lambda_i)^{\otimes\mu_i} \ , \tag{3.3}$$

with all $\Lambda_i\in F$, are easily decomposed using the Pieri formula (3.1). The decompositions are triangular in the irreducible highest-weight modules $M(\sigma)$, $\sigma\in P_\geq$:

$$M(\Lambda^\lambda) \ = \bigoplus_{P_\geq\ni\sigma\leq\lambda} \Omega_{\lambda,\sigma}\, M(\sigma) \ . \tag{3.4}$$

Here $\sigma\leq\lambda$ means that $\lambda-\sigma$ is a non-negative integer linear combination of positive roots, i.e. $\lambda-\sigma\in\mathbb{Z}_\geq R_>$. So $\Omega_{\lambda,\sigma}$ is a triangular matrix.

The polynomial deformation of this, encoding the threshold levels, is just

$$\mathcal{L}^t\left[M(\Lambda^\lambda)\right] \;=\; \bigoplus_{P_\geq \ni \sigma \leq \lambda} \Omega_{\lambda,\sigma}[\ell]\, M(\sigma) \quad . \tag{3.5}$$

Some A_2 examples (to be used shortly) will make this clear. We find:

$$\mathcal{S}_t\left[M(\Lambda^{(2,2)})\right] \;=\; M(2,2)_4 \oplus M(3,0)_3 \oplus 4M(1,1)_{2,2,2,3} \oplus M(0,3)_3 \oplus 2M(0,0)_{1,2} \quad , \tag{3.6}$$

or

$$\begin{aligned}\mathcal{L}^t\left[M(\Lambda^{(2,2)})\right] \;=\; & \ell^4\, M(2,2) \oplus \ell^3\, M(3,0) \oplus (\ell^3 + 3\ell^2)\, M(1,1) \\ & \oplus \ell^3\, M(0,3) \oplus (\ell + \ell^2)\, M(0,0)\ ;\end{aligned} \tag{3.7}$$

and

$$\mathcal{S}_t\left[M(\Lambda^{(1,1)})\right] \;=\; M(1,1)_2 \oplus M(0,0)_1\ , \quad \mathcal{S}_t\left[M(\Lambda^{(0,0)})\right] \;=\; M(0,0)_0 \quad , \tag{3.8}$$

or

$$\mathcal{L}^t\left[M(\Lambda^{(1,1)})\right] \;=\; \ell^2\, M(1,1) \oplus \ell^1\, M(0,0)\ , \quad \mathcal{L}^t\left[M(\Lambda^{(0,0)})\right] \;=\; M(0,0) \quad . \tag{3.9}$$

From (3.4), the highest-weight modules can be expressed as polynomials in the fundamental weights:

$$M(\sigma) \;=\; \bigoplus_{P_\geq \ni \mu \leq \sigma} (\Omega^{-1})_{\sigma,\mu}\, M(\Lambda^\mu) \quad . \tag{3.10}$$

That is, $M(\sigma)$ can be written as a direct sum of fundamental monomials $M(\Lambda^\mu)$. This is the Giambelli formula, in the non-determinantal form that can be applied to all simple Lie algebras, not just A_r. Notice that the inversion of Ω is greatly simplified by its triangularity, and its inverse is also triangular. A simple A_2 example of (3.10) is

$$M(1,1) \;=\; M(\Lambda^{(1,1)}) \ominus M(\Lambda^{(0,0)}) \quad . \tag{3.11}$$

The characters of X_r form an algebra with structure constants equal to the tensor product coefficients. The Giambelli formula (3.10) gives rise to a polynomial realisation of this character algebra. One simply replaces $M(\Lambda^\mu)$ with $\prod_{i=1}^r x_i^{\mu_i} =: x^\mu$. The resulting polynomial

$$S_\sigma(x) \;:=\; \sum_{P_\geq \ni \mu \leq \sigma} (\Omega^{-1})_{\sigma,\mu}\, x^\mu \quad , \tag{3.12}$$

is known as a Schur polynomial, a type of Schubert polynomial [22]. For example, x_1x_2-1 is the Schur polynomial of the A_2 module $M(1,1)$, by (3.11). With the addition and subtraction of polynomials, the character algebra extends to a ring.

Is there a useful threshold-level version of the Giambelli formula? The inverse $\Omega^{-1}[\ell]$ of the matrix $\Omega[\ell]$ in (3.5) has entries that are negative powers of ℓ. These are difficult to interpret in the context of threshold level. We conclude that the normal, ℓ-independent matrix Ω^{-1} should be used. We can write useful formulas for the threshold polynomials in terms of Ω^{-1}, and its deformed inverse $\Omega[\ell]$:

$$T_{\lambda,\mu}^{\nu}[\ell] \;=\; \ell^{(\lambda,\theta)}\circ\ell^{(\mu,\theta)}\circ\sum_{\alpha,\beta\in P_{\geq}}(\Omega^{-1})_{\lambda,\alpha}\,(\Omega^{-1})_{\mu,\beta}\,(\Omega)_{\alpha+\beta,\nu}[\ell]\quad . \tag{3.13}$$

We'll illustrate this formula on the A_2 example with $M(\lambda)=M(\mu)=M(1,1)$, using the subscript notation. First, the required matrix elements of Ω^{-1} are provided by

$$\begin{aligned} M(1,1)^{\otimes 2} \;&=\; \Big[M(\Lambda^{(1,1)})\ominus M(\Lambda^{(0,0)})\Big]^{\otimes 2}\\ &= M(\Lambda^{(2,2)})\ominus 2M(\Lambda^{(1,1)})\oplus M(\Lambda^{(0,0)})\quad . \end{aligned} \tag{3.14}$$

Substituting the fundamental monomials (3.6) and (3.8), described by $\Omega[\ell]$, we get

$$\begin{aligned} \mathcal{L}^t\left[M(1,1)^{\otimes 2}\right] \;&=\ell^2\circ\Big\{\,\ell^4M(2,2)\oplus\ell^3M(3,0)\oplus(3\ell^2+\ell^3)M(1,1)\oplus\ell^3M(0,3)\\ &\qquad\oplus(\ell+\ell^2)M(0,0)\ominus 2\left[\ell^2M(1,1)\oplus\ell M(0,0)\right]\,\oplus M(0,0)\,\Big\}\\ &=\ell^4M(2,2)\oplus\ell^3M(3,0)\oplus\ell^3M(0,3)\oplus(\ell^2+\ell^3)M(1,1)\oplus\ell^2M(0,0)\quad , \end{aligned} \tag{3.15}$$

the correct result.

The deformed Pieri formula (3.2) makes straightforward the calculation of decompositions involving fundamental monomials, like $M(\Lambda^\beta)$. We write

$$\mathcal{L}^t\left[M(\lambda)\,\otimes\,M(\Lambda^\beta)\right] \;=\; \bigoplus_{\nu\in P_{\geq}} T_{\lambda,\Lambda^\beta}^{\nu}[\ell]\,M(\nu)\;. \tag{3.16}$$

Then the threshold polynomials may also be calculated from the simpler polynomials $T_{\lambda,\Lambda^\beta}^{\nu}[\ell]$:

$$T_{\lambda,\mu}^{\nu}[\ell] \;=\; \ell^{(\mu,\theta)}\,\circ\,\sum_{\beta\in P_{\geq}}(\Omega^{-1})_{\mu,\beta}\,T_{\lambda,\Lambda^\beta}^{\nu}[\ell]\quad . \tag{3.17}$$

Using (3.11), an A_2 example is

$$\begin{aligned}\mathcal{L}^t\left[M(1,1)^{\otimes 2}\right] &= \ell^2\circ\mathcal{L}^t\Big[\,M(1,1)\otimes\big(M(\Lambda^{(1,1)})\ominus M(\Lambda^{(0,0)})\big)\,\Big]\\ &= \ell^4 M(2,2)\oplus\ell^3 M(3,0)\oplus(2\ell^2+\ell^3)M(1,1)\oplus\ell^3 M(0,3)\oplus\ell^2 M(1,1)\\ &\quad\ominus\ell^2 M(1,1)\ \ .\end{aligned} \tag{3.18}$$

This is again the correct result (see (3.15)).

Finally, we can also multiply two Schur polynomials for $M(\lambda)$ and $M(\mu)$ together, using the coefficients $T^{\nu}_{\Lambda^\alpha,\Lambda^\beta}$, defined in the obvious way:

$$T^{\nu}_{\lambda,\mu}[\ell] \;=\; \ell^{(\lambda,\theta)}\circ\ell^{(\mu,\theta)}\circ\sum_{\alpha,\beta\in P_{\geq}}(\Omega^{-1})_{\lambda,\alpha}\;(\Omega^{-1})_{\mu,\beta}\,T^{\nu}_{\Lambda^\alpha,\Lambda^\beta}[\ell]\ \ . \tag{3.19}$$

To conclude this section, we note that in our deformed Schubert calculus, there is no analogue of the fusion potential that was so important in [1]. We argue that a deformed potential that incorporates the threshold levels cannot be written. Gepner could write a fusion potential because at fixed level k, the fusion rules are truncations of the tensor product rules. The truncated parts can be set to zero by fusion constraints, that can be derived from the potential. On the other hand, when the threshold level is incorporated into a tensor product, as in (3.5) vs. (3.4), there is no truncation. Instead of constraints, one could only hope to find replacements that would change the right-hand side of (3.4) into that of (3.5), for example. But that is exactly what we do: (3.5) is obtained from (3.4) by replacing Ω with $\Omega[\ell]$. A minimal set of such replacements would be those obtained by replacing the right-hand side of the undeformed Pieri rule (3.1) with that of the deformed one (3.2).

Incidentally, we have seen that the Pieri rule with threshold level (3.2) contains the same information as the fusion potentials of Gepner, for all (non-negative integer) levels. So does the generating function for the fusion potentials [1]. It might be interesting to make this more precise.

4. Deformed tensor product coefficients and weight multiplicities

The threshold polynomials (2.17) and (2.21) are deformations of the tensor product coefficients and affine (WZW) fusion coefficients, respectively. It is interesting to compare them with the q-deformations of these objects studied previously.

WZW fusion coefficients are alternating affine-Weyl (W^k) sums of tensor product coefficients [23][24][25][26]. In [14] (see also [15]), the corresponding q-fusion coefficients (for affine A_r) are defined in similar fashion in terms of the q-tensor product coefficients [10][11][12][13]. Since the ordinary (undeformed) tensor product coefficients are also alternating Weyl sums of the weight multiplicities of Lie algebras, the fusion coefficients can also be expressed in that way. In the q-deformed case, the tensor product and fusion coefficients are related to Lusztig's q-deformed weight multiplicities [16], in turn related to the famous Kazhdan-Lusztig polynomials [27].

Let us start with an example, taken from [14]. They find, for the q-deformation of the A_3 tensor product $M(1,1,0)^{\otimes 3}$, the following decomposition:

$$\begin{aligned}(q^3{+}q^6)\, M(0,0,3) &\oplus (2q^4+3q^5+2q^6+q^7)\, M(0,1,1) \oplus (q^2+2q^3+q^4)\, M(0,3,1)\\ &\oplus (q^5+2q^6+q^7)\, M(1,0,0) \oplus (q^2+2q^3+3q^4+2q^5)\, M(1,1,2)\\ &\oplus (2q^3+3q^4+3q^5+q^6)\, M(1,2,0) \oplus (q+q^2)\, M(1,4,0)\\ &\oplus (q^3+3q^4+3q^5+2q^6)\, M(2,0,1) \oplus (q+2q^2+2q^3+q^4)\, M(2,2,1)\\ &\oplus (q^2+2q^3+q^4)\, M(3,0,2) \oplus (q^2+2q^3+2q^4+q^5)\, M(3,1,0)\\ &\oplus\ M(3,3,0) \oplus (q+q^2)\, M(4,1,1) \oplus (q^3)\, M(5,0,0)\ .\end{aligned} \tag{4.1}$$

This should be compared with the threshold-level version of the same tensor product:

$$\begin{aligned}\mathcal{L}^t\left[M(1,1,0)^{\otimes 3}\right]\ &= (2\ell^3)\, M(0,0,3) \oplus (\ell^2+7\ell^3)\, M(0,1,1) \oplus (4\ell^4)\, M(0,3,1)\\ &\oplus (2\ell^2+2\ell^3)\, M(1,0,0) \oplus (8\ell^4)\, M(1,1,2)\\ &\oplus (4\ell^3+5\ell^5)\, M(1,2,0) \oplus (2\ell^5)\, M(1,4,0)\\ &\oplus (5\ell^3+4\ell^4)\, M(2,0,1) \oplus (6\ell^5)\, M(2,2,1)\\ &\oplus (4\ell^5)\, M(3,0,2) \oplus (4\ell^4+2\ell^5)\, M(3,1,0)\\ &\oplus (\ell^6)\, M(3,3,0) \oplus (2\ell^6)\, M(4,1,1) \oplus (\ell^5)\, M(5,0,0)\ .\end{aligned} \tag{4.2}$$

From this example, we see no clear relation between the q-deformations and the ℓ-deformations, except that they coincide at $q=\ell=1$.

In order to define the q-tensor product coefficients, one introduces the q-deformed Kostant partition function $K(\beta;q)$:

$$\prod_{\alpha\in R_>} (1-qe^{\alpha})^{-1}\ \ =:\ \sum_{\beta\in\mathbb{Z}_{\geq}R_>} K(\beta;q)e^{\beta}\quad . \tag{4.3}$$

From this we see that the powers of q count the number of positive roots in a decomposition of an element of $\mathbb{Z}_{\geq}R_{>}$. The q-deformed weight multiplicities are

$$\mathrm{mult}_\lambda(\mu;q) \ := \ \sum_{w\in W} (\det w)\, K(w.\lambda-\mu;q) \ , \tag{4.4}$$

and we get the q-deformed tensor product coefficients as

$$T^{\nu}_{\lambda,\mu}(q) \ := \ \sum_{w\in W} (\det w)\, \mathrm{mult}_\mu(w.\nu-\lambda;q) \ . \tag{4.5}$$

Notice we use different brackets to distinguish the different deformations: $T^{\nu}_{\lambda,\mu}(q)$ vs. $T^{\nu}_{\lambda,\mu}[\ell]$. Finally, the q-fusion coefficients [14] can be found from

$${}^{(k)}T^{\nu}_{\lambda,\mu}(q) \ := \ \sum_{w\in W^k} (\det w)\, T^{w.\nu}_{\lambda,\mu}(q) \ . \tag{4.6}$$

It would be interesting to define the threshold polynomial versions of the q-Kostant partition function, and the q-multiplicities. The relation between the q-tensor product coefficients and the threshold polynomials might then be extracted. We have not succeeded in finding the "ℓ-Kostant partition function". But the ℓ-multiplicities may be defined using a conjectural refinement [9][17] of the Gepner-Witten depth rule [7]:

$${}^{(k)}T^{\nu}_{\lambda,\mu} \ = \ \dim\left\{\, v\in M(\mu;\nu-\lambda)\,|\,(f_i)^{\nu_i+1}v=0,\ \forall i\in\{0,1,\ldots,r\}\,\right\} \ . \tag{4.7}$$

Here $M(\mu;\nu-\lambda)$ is the subspace of weight $\nu-\lambda$ of the module $M(\mu)$, so that $\dim M(\mu;\sigma)=\mathrm{mult}_\mu(\sigma)$. The f_i are the lowering operators corresponding to the simple roots of the simple Lie algebra $X_r\subset X_{r,k}$, for $i=1,\ldots,r$. f_0 is identified with e_θ, the raising operator corresponding to the highest root θ of X_r. Recall that the *depth* of a vector v (in a module $M(\mu)$, say) is defined as the non-negative integer d such that $(e_\theta)^{\,d}\,v\neq 0$, while $(e_\theta)^{\,d+1}\,v=0$. The relation between (4.7) and the Gepner-Witten depth rule is then clear.

By (4.7), we see that the subspaces

$$M(\mu;\sigma|d) \ := \ \{\, v\in M(\mu;\sigma)\ |\ (e_\theta)^{\,1+d}\,v=0,\ \ (e_\theta)^{d}\,v\neq 0\,\} \tag{4.8}$$

of the spaces $M(\mu;\sigma)$ are relevant. Because of their relation to the depth, the multiplicities that are the dimensions of these spaces,

$$\mathrm{mult}_\mu(\sigma|d) \ := \ \dim M(\mu;\sigma|d) \ , \tag{4.9}$$

were dubbed “profundities” in [18]. They have the same relation to the threshold coefficients (see (2.12)) that the the usual multiplicities have to the tensor product coefficients:

$$T^{\nu(t)}_{\lambda,\mu} \;=\; \sum_{w\in W} (\det w)\, \mathrm{mult}_\mu\, (w.\nu-\lambda \,|\, t-(\nu,\theta)) \;\;, \tag{4.10}$$

where $t \geq (\nu,\theta)$. Substituting this into (2.17) gives

$$T^{\nu}_{\lambda,\mu}[\ell] \;:=\; \ell^{\,(\nu,\theta)} \sum_{w\in W} (\det w)\, \mathrm{mult}_\mu(w.\nu-\lambda;\ell) \;, \tag{4.11}$$

with ℓ-deformed multiplicities

$$\mathrm{mult}_\mu(\sigma;\ell) \;:=\; \sum_{d\in\mathbb{Z}_\geq} \ell^{\,d}\, \mathrm{mult}_\mu(\sigma|d) \;. \tag{4.12}$$

So the only complication is the overall factor $\ell^{\,(\nu,\theta)}$, and the ℓ-deformed multiplicities are generating functions for the profundities.

Incidentally, in deriving this last result, we used the relation

$$d \;=\; t \;-\; (\nu,\theta) \tag{4.13}$$

between the depth d and the threshold level t of a fixed “coupling”. Notice that this is identical to (2.15). Hence the threshold delay of a coupling equals the depth of the corresponding vector in (4.7).

5. Fusion paths and the threshold level

Paths on fusion graphs are important in certain integrable lattice models that are related to conformal field theory (see [19][20], and references therein). These graphs may also have a more fundamental significance, indicated by the correspondence between A_1 modular invariants and the $A-D-E$ graphs [28].

If we restrict to the case $X_{r,k} = A_{r,k}$, then the points of the relevant graphs correspond to the weights of $P^k_\geq$. There is a distinct directed graph ${}^{(k)}\mathcal{G}_i$ for each fundamental weight $\Lambda_i \in F$. The edges of the graph ${}^{(k)}\mathcal{G}_i$ are encoded in its incidence matrix ${}^{(k)}G_i$, which is not necessarily symmetric. $({}^{(k)}G_i)_{\lambda,\mu}$ is the number of edges joining node λ with node μ. The fusion graph is defined by $({}^{(k)}G_i)_{\lambda,\mu} = {}^{(k)}T^{\nu}_{\Lambda_i,\lambda}$, or ${}^{(k)}G_i = {}^{(k)}T_{\Lambda_i}$, hence the name. One can also define a graph ${}^{(k)}\mathcal{G}$ with incidence matrix ${}^{(k)}G := \sum_{i=1}^r {}^{(k)}G_i$.

A fusion path is a path on a fusion graph. Such paths parametrise the Hilbert space of certain integrable two-dimensional lattice models. The basic construction is a representation of a (quotient of a) Hecke algebra on this space. It guarantees that the models' Boltzmann weights satisfy the Yang-Baxter equation, ensuring integrability.

Due to (2.9),(2.10), and since $P_\geq^k \subset P_\geq^{k+1} \subset \cdots \subset P_\geq$, we can think of the graphs ${}^{(k)}\mathcal{G}_i$ and ${}^{(k)}\mathcal{G}$ in the infinite-level limit as tensor product graphs. (Here we restrict consideration to paths involving weights that do not increase with the level.) Such paths on $P_\geq$ will also be paths on all ${}^{(k)}\mathcal{G}$, for all levels k greater than a certain threshold level t. This threshold level, is just the maximum height $ht(\lambda) := (\lambda, \theta)$ of the weights $\lambda \in P_\geq$ on the path.

Key to the modified Schubert calculus described above were the fundamental monomials, and their decompositions (3.4). But the fundamental monomials $M(\Lambda^\mu)$ generate paths in $P_\geq$: to every module $M(\sigma)$ in the decomposition (3.4) there corresponds a path on $P_\geq$ that begins at the weight 0 and ends at λ. To each factor $M(\Lambda)$, $\Lambda \in F$, in the monomial corresponds a segment of the path that connects nodes of the graph that differ by some $\varphi \in P(\Lambda)$.

The threshold level is included by modifying (3.4) to (3.5), using the ℓ-Pieri rule. From (3.2), we see that the threshold level is the maximum height of a path on the (infinite) tensor product graph of A_r. This is the main point of this section.

We should emphasise, however, that the correspondence between the fundamental monomials and tensor product paths is not one-to-one. The polynomial realisation (3.12) of the fusion ring is possible because $M(\Lambda) \otimes M(\Lambda') = M(\Lambda') \otimes M(\Lambda)$, for all $\Lambda, \Lambda' \in F$. But the order of tensor product factors $M(\Lambda)$ changes the path. This can be made clear by writing a generating matrix for fundamental monomials:

$$\Phi \ := \sum_{\mu \in P_\geq} e^\mu \, T_{\Lambda_1}^{\mu_1} \cdots T_{\Lambda_r}^{\mu_r} \ = \ \prod_{i=1}^{r} \left[1 - e^{\Lambda_i} \, T_{\Lambda_i} \right]^{-1} \ . \tag{5.1}$$

Here e^μ denotes a formal exponential, satisfying $e^\mu e^\nu = e^{\mu+\nu}$. $\Phi_{\lambda,\mu}$ will equal the sum over all fundamental monomials that when tensored with $M(\lambda)$, include $M(\mu)$ in the decomposition, multiplied by the formal exponential of the monomial weight of each. Putting $e^\mu \to 1$ then gives the number of such monomials. On the other hand, to generate all paths connecting nodes λ and μ, we need $\Theta_{\lambda,\mu}$ instead, where

$$\Theta \ := \ \left[1 - \sum_{i=1}^{r} e^{\Lambda_i} T_{\Lambda_i} \right]^{-1} \ . \tag{5.2}$$

The deformations of these two generating matrices are simple to write. One only needs to replace the tensor product matrices T_{Λ_i} with their ℓ-deformations, and insist that they are multiplied in the manner of (2.32). So we get

$$\Phi[\ell] \;=\; \prod_{i=1}^{r} \left\{1 - e^{\Lambda_i}\, T_{\Lambda_i}[\ell]\right\}^{(\circ\, -1)} \ , \tag{5.3}$$

where the notation (we hope) is clear, and a similar formula analogous to (5.2).

6. Conclusion

Our main result is a Schubert-type calculus for affine fusion that incorporates the threshold level. At fixed level, fusion constraints are natural because a fusion rule is a truncation of a tensor product decomposition. Thus fusion potentials that generate constraints are possible, if not necessary. On the other hand, in the threshold level formalism one doesn't truncate a tensor product, but rather replaces it with a deformed version. So, instead of using a fusion potential to generate constraints, one just deforms the tensor products and then all (non-negative integer) levels are treated on equal footing. The deformations are generated by the deformed version of the Pieri rule, (3.2).

In summary then, to include the threshold levels in a calculus of Schubert type, use the undeformed Giambelli formula (3.10), and the deformed Pieri formula (3.2). Then the threshold polynomials can be calculated by (3.13),(3.17), or (3.19).

Another result is the comparison in section 4 of the threshold polynomials with the q-deformed tensor product and fusion coefficients. In particular, we found the analogue of the q-deformed weight multiplicities in our ℓ-deformation.

We also discussed the interpretation of the threshold level for the decomposition of fundamental monomials, as in (3.5). In the corresponding path on a tensor product graph, the threshold level is just the maximum height of weights on that path.

To close, let us mention a few possible directions from this work.

One could hope to make the connection between the q-deformations and ℓ-deformations more precise, extending our section 4. We also expect that one could define a q-Schubert calculus for the q-tensor product coefficients (4.5), in a straightforward way. In contrast with the ℓ-deformed case, both $\Omega(q)$ and $\Omega^{-1}(q)$ should be important. It might be of interest to introduce q-analogues of the fusion constraints and potentials of Gepner's calculus, for the q-fusion-coefficients.

The ℓ-deformed Schubert calculus is relevant to the search for a Littlewood-Richardson rule for affine fusion [29]. In the present context, the usual Littlewood-Richardson rule for tensor products is related to (3.17), at $\ell = 1$. This formula involves the tensor product of two modules $M(\lambda)$, $M(\mu)$, where one is expressed in terms of fundamental monomials by (3.10): $M(\mu) = \sum_\beta (\Omega)^{-1}_{\mu,\beta} M(\Lambda^\beta)$. The rule gives a way of avoiding the cancellations inherent in (3.17) (see (3.15), e.g.). It identifies a choice of a part of the decompositions of the $M(\Lambda^\beta)$ in (3.17) that leads directly to the result. Unfortunately, the deformed Pieri rule applied to that choice gives incorrect threshold levels (one finds $2M(1,1)_{2,2}$ instead of the $2M(1,1)_{2,3}$ of (3.15), e.g.). Calculations of the type (3.17), however, show us all the parts. And so we can hope that more in-depth analysis will reveal the appropriate modification.

Finally, if it exists, a motivation other than computational for the threshold level should be found. It might be revealed by finding the meaning of the threshold level in the many different physical and mathematical realisations of affine fusion.

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References

[1] D. Gepner, Commun. Math. Phys. **141** (1990) 381

[2] H. Hiller, Geometry of Coxeter groups (Pitman, 1982);
W. Fulton, Young tableaux (Cambridge University Press, 1997), Part III;
P. Griffiths, J. Harris, Principles of algebraic geometry (Wiley, 1978)

[3] E. Witten, in: Geometry, Topology and Physics; Conf. Proc. and Lecture Notes in Geom. Topol. **VI** (1995) 357

[4] K. Intriligator, Mod. Phys. Lett. **A6** (1991) 3543

[5] C. Vafa, in: Essays on Mirror Manifolds, ed. S.-T. Yau (International Press, 1992)

[6] D. Nemeschansky, N. Warner, Nucl. Phys. **B380** (1992) 241

[7] D. Gepner, E. Witten, Nucl. Phys. **B278** (1986) 493

[8] C.J. Cummins, P. Mathieu, M.A. Walton, Phys. Lett. **254B** (1991) 386

[9] A.N. Kirillov, P. Mathieu, D. Sénéchal, M.A. Walton, Nucl. Phys. **B391** (1993) 651

[10] A. Schilling, S.O. Warnaar, Commun. Math. Phys. **202** (1999) 359

[11] A.N. Kirillov, M. Shimozono, A generalization of the Kostka-Foulkes polynomials, math.QA/9803062

[12] M. Shimozono, J. Weyman, Graded characters of modules supported by the closure of a nilpotent conjugacy class, math.QA/9804036

[13] B. Leclerc, J-Y. Thibon, Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials, math.QA/9809122

[14] O. Foda, B. Leclerc, M. Okado, J-Y. Thibon, Ribbon tableaux and q-analogues of fusion rules in WZW conformal field theories, math.QA/9810008

[15] A. Schilling, M. Shimozono, Bosonic formula for level-restricted paths, math.QA/9812106

[16] G. Lusztig, Astérisque **101- 102** (1983) 208-229

[17] A.N. Kirillov, P. Mathieu, D. Sénéchal, M.A. Walton, in: Group-Theoretical Methods in Physics, Proceedings of the XIXth International Colloquium, Salamanca, Spain, 1992, Vol. 1 (CIEMAT, Madrid, 1993)

[18] M.A. Walton, Can. J. Phys. **72** (1994) 527

[19] J.-B. Zuber, Commun. Math. Phys. **179** (1996) 265

[20] V.B. Petkova, J.-B. Zuber, Nucl. Phys. **B463** (1996) 161

[21] W. Fulton, J. Harris, Representation theory: a first course (Springer-Verlag, 1991)

[22] I. G. Macdonald, Notes on Schubert polynomials (Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montréal, 1991)

[23] V.G. Kac, Infinite dimensional Lie algebras, 3rd ed. (Cambridge U. Press, 1990)

[24] M.A. Walton, Phys. Lett. **241B** (1990) 365; Nucl. Phys. **B340** (1990) 777

[25] P. Furlan, A. Ganchev, V. Petkova, Nucl. Phys. **B343** (1990) 205

[26] F. Goodman, H. Wenzl, Adv. Math. **82** (1990) 244

[27] D. Kazhdan, G. Lusztig, Invent. Math. **53** (1979) 165

[28] A. Cappelli, C. Itzykson, J.-B. Zuber, Commun. Math. Phys. **113** (1987) 1

[29] M.A. Walton, J. Math. Phys. **39** (1998) 665